Functional Analysis

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Lecture 7

Orthogonal projection

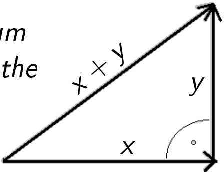
math.uwb.edu.pl/∼zaf/kwasniewski/teaching

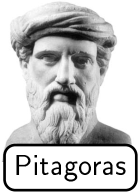
H – inner product space.

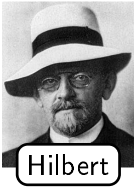
Def. Vectors $x, y \in H$ are **orthogonal** (**perpendicular** to each other), if $\langle x, y \rangle = 0$. We then write $x \perp y$.

Rem. $x \perp y \iff y \perp x$. And $\forall_{y \in H}\ x \perp y \iff x \perp x \iff x = 0$.

In a right angled triangle, the sum of the squares of the lengths of the legs is equal to the square of the length of the hypotenuse!







[Pythagoras' theorem]

If $x \perp y$, then $\|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Proof: $\|x + y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y \rangle + \|y\|^2$

Rem. If $\mathbb{F} = \mathbb{R}$, then

$x \perp y \iff \|x + y\|^2 = \|x\|^2 + \|y\|^2$.

Def. The **orthogonal complement** of the set $M \subseteq H$ is

$$M^{\perp} := \{x \in H : x \perp y \text{ dla każdego } y \in M\}.$$

If $x \in M^{\perp}$, then we also write $x \perp M$. For two sets $N, M \subseteq H$ we write $N \perp M$ if $n \perp m$ for all $n \in N$ and $m \in M$.

Prop. $M^{\perp}$ is a closed linear space.

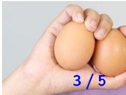
Proof: Every $y \in H$ defines a linear functional $f_y : H \to \mathbb{F} = \mathbb{R}, \mathbb{C}$

$$f_y(x) := \langle x, y \rangle, \qquad x \in H,$$

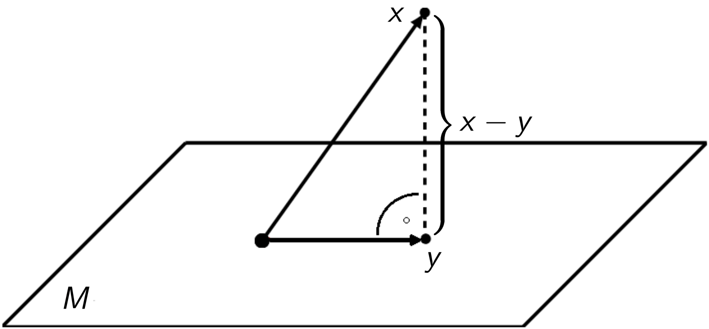
By the Schwartz inequality $|f_y(x)| = |\langle x, y \rangle| \leqslant \|x\| \cdot \|y\|$. Hence f_y is bounded and $\|f_y\| \leqslant \|y\|$ (in fact $\|f_y\| = \|y\|$). Thus f_y is continuous. Accordingly, the kernel $\ker f_y = f_y^{-1}(0) \subseteq H$ is a closed linear subspace. Since $M^{\perp}$ is the intersection of such kernels:

$$M^{\perp} = \{x \in H : f_y(x) = 0 \text{ for every } y \in M\} = \bigcap_{y \in M} \ker f_y$$

it is a closed linear subspace. ■



Def. The **orthogonal projection of a vector** $x \in H$ onto a subspace $M \subseteq H$ is a vector $y \in M$ such that $x - y \perp M$.



We write then $P_M x = y$. Hence

$$P_M x = y \quad \overset{def}{\Longleftrightarrow} \quad y \in M \text{ and } \forall_{z \in M} \; \langle x - y, z \rangle = 0.$$

The orthogonal projection $P_M x$ is uniquely determined! 

Problem: Does orthogonal projection always exist?

Ex. (Orthogonal projection onto a one-dimensional subspace)

Let $M = \{\lambda y : \lambda \in \mathbb{F}\}$, where $y \in H \setminus \{0\}$, and let $x \in H$. Then $P_M x = \lambda_0 y$, where $\lambda_0 \in \mathbb{F}$ is such that $x - \lambda_0 y \perp M$. What is λ_0?

$$
\begin{aligned}
x - \lambda_0 y \perp M &\iff \forall_{z \in M} \ \langle x - \lambda_0 y, z \rangle = 0 \\
&\iff \forall_{\lambda \in \mathbb{F}} \ \langle x - \lambda_0 y, \lambda y \rangle = 0 \\
&\iff \forall_{\lambda \in \mathbb{F}} \ \overline{\lambda} \cdot \langle x - \lambda_0 y, y \rangle = 0 \\
&\iff \langle x - \lambda_0 y, y \rangle = 0 \iff \langle x, y \rangle - \lambda_0 \langle y, y \rangle = 0 \\
&\iff \lambda_0 = \tfrac{\langle x, y \rangle}{\langle y, y \rangle}.
\end{aligned}
$$

Hence

$$P_M x = \lambda_0 y = \tfrac{\langle x, y \rangle}{\langle y, y \rangle} y = \tfrac{\langle x, y \rangle}{\|y\|^2} y.$$

This is sometimes called the **orthogonal projection** of vector x on **vector** y and is denoted by $P_y x$.

Rem. Calculating $\|x - P_y x\|$, the distance from x to the subspace $M = \{\lambda y : \lambda \in \mathbb{F}\}$, one obtains a proof of the Schwartz inequality.